ISRAEL JOURNAL OF MATHEMATICS **164** (2008), 125–152
DOI: 10.1007/s11856-008-0023-4

MANY SOLUTIONS FOR ELLIPTIC EQUATIONS WITH CRITICAL EXPONENTS

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ABSTRACT

Let Ω be an open bounded domain in $\mathbb{R}^N (N \geq 3)$ and $2^* = \frac{2N}{N-2}$. We are concerned with two kinds of critical elliptic problems. The first one is

$$(*) \qquad -\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{m-2}u + \theta |u|^{2^*-2}u \quad u \in H_0^1(\Omega),$$

where $0 \in \Omega$, $0 < \mu < (\frac{N-2}{2})^2$, $2 < m < 2^*$ and $\lambda > 0$. By using the fountain theorem and concentration estimates, if $N \geq 7$ and $\theta > 0$, we establish the existence of infinitely many solutions for the following regularization of $(*)$ with small number $\epsilon > 0$

$$-\Delta u - \mu \frac{u}{|x|^2 + \epsilon} = \lambda u + |u|^{m-2}u + \theta |u|^{2^*-2}u \quad u \in H_0^1(\Omega).$$

Then if $\theta > 0$ is suitably small, we obtain many solutions for problem $(*)$ by taking the process of approximation.

The second problem is

$$-\Delta u = |u|^{2^*-2}u + t|u|^{q-1}u \quad u \in H_0^1(\Omega),$$

where $q \in (0,1)$, $t > 0$. By using similar methods as in $(*)$, we prove that if $N \geq 7$, $\frac{4}{N-2} < q < 1$ and $t > 0$, there exist infinitely many solutions with positive energy. In particular, we give a positive answer to one open problem proposed by Ambrosetti, Brezis and Cerami [1].

Received May 12, 2006 and in revised from September 26, 2006

1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $0 \in \Omega$. We are concerned with the following problem

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{m-2}u + \theta |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < \mu < \bar{\mu} = ((N-2)/2)^2$, $2 < m < 2^*$, $\theta > 0$, $\lambda > 0$ and $2^* = 2N/(N-2)$ is the critical exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^t(\Omega)$.

Definition 1.1: $u \in H_0^1(\Omega)$ is said to be a weak solution of problem (1.1) if u satisfies

$$\int_\Omega \Big(\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} - \lambda uv - |u|^{m-2}uv - \theta |u|^{2^*-2}uv \Big) dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

By standard elliptic regularity arguments, we have $u \in C^2(\Omega \backslash \{0\}) \cap C^1(\overline{\Omega} \backslash \{0\})$. It is well-known that the nontrivial solutions of problem (1.1) are equivalent to the nonzero critical points of the energy functional

$$I_{\theta,0}(u) = \frac{1}{2} \int_\Omega \Big(|\nabla u|^2 - \frac{\mu |u|^2}{|x|^2} - \lambda |u|^2 \Big) dx - \frac{1}{m} \int_\Omega |u|^m dx - \frac{\theta}{2^*} \int_\Omega |u|^{2^*} dx \qquad u \in H_0^1(\Omega).$$

There are many papers which are significantly related with problem (1.1). For examples, S. Terracini [20] considered the nonlinear problem in $\mathbb{R}^N \backslash \{0\}$:

$$-\Delta u = a(x/|x|)u/|x|^2 + f(x,u),$$

where $a \in C^1(S^{N-1}, \mathbb{R})$ and f is a superlinear function. In [20], the diverging Palais–Smale sequences are analyzed, multiplicity result and the uniqueness (modulo rescaling) of positive solutions are established by means of variational methods together with sophisticated versions of the moving plane method of Aleksandrov.

E. Jannelli [16] considered the following problem:

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

and established Brezis–Nirenberg type results. D. Cao and P. Han [8] proved that if $\mu \in [0, \bar{\mu} - (\frac{N+2}{N})^2)$, then problem (1.2) admits a nontrivial solution for all $\lambda > 0$. A. Ferrero and F. Gazzola [13] considered a more general form of (1.2) and obtained some results. Replacing the nonlinearities in (1.1) with $f(x,u)$ (here $f(x,\cdot)$ is superlinear at zero and subcritical at infinity), M. Schechter and W. Zou [18] proved the existence of infinitely many sign-changing solutions under some conditions. Other relevant papers on this matter see [6, 9, 10, 12, 15] and the references cited therein.

One important question on problem (1.1) is that for any $\mu \in (0,\bar{\mu})$, whether there exist infinitely many solutions of (1.1) with every $\lambda > 0$. As far as we know, there are little results on this question. In this paper, we prove that if $\theta > 0$ is suitably small, problem (1.1) has many solutions.

Let X be a Banach space. The functional $J \in C^1(X,\mathbb{R})$ is said to satisfy the $(P.S.)_c$ condition if any sequence $\{u_n\} \subset X$ such that as $n \to \infty$

$$J(u_n) \to c,\ dJ(u_n) \to 0 \quad \text{strongly in } X^*$$

contains a subsequence converging in X to a critical point of J.

Note that the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact, which leads to that the functional $I_{\theta,0}$ does not satisfy $(P.S.)_c$ condition for any $c > 0$. Since we are only interested in solutions with positive energy, fountain theorem and its dual theorem (see [4, 21]) are not directly applicable to the case of (1.1).

G. Devillanova and S. Solimini [11] recently considered problem (1.2) with $\mu = 0$ and obtained infinitely many solutions for any $\lambda > 0$. Two important methods employed in [11] are concentration estimates and the lower bound of the augmented Morse index on min-max points (see [3]), which seem not applicable to problem (1.1). Since for $\mu \in (0,\bar{\mu})$, nontrivial solutions of problem (1.1) have singularity at the origin (see Theorem 1.1 below), we cannot establish the uniform bound through concentration estimates and the lower bound of the augmented Morse index for solutions of (1.1) as in [11]. In order to overcome these difficulties, we have to look for other methods to deal with problem (1.1).

The first result of this paper is on the asymptotic behavior at the origin for nontrivial solutions of the following problem with small number $\epsilon \geq 0$, $\theta > 0$

$$(1.3) \qquad \begin{cases} -\Delta u - \mu \frac{u}{|x|^{2+\epsilon}} = \lambda u + |u|^{m-2}u + \theta |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p \in [2, 2^*]$. That is,

THEOREM 1.1: *Assume that* $N \geq 3$, $p \in [2, 2^*]$, $m \in (2, 2^*)$, $\mu \in (0, \bar{\mu})$, $\theta > 0$ *and* $\lambda > 0$. *If each solution* $u_{\theta,\epsilon} \in H_0^1(\Omega)$ *of problem* (1.3) *with* $\epsilon > 0$ *satisfies that* $\theta^{1/2^*}\|u_{\theta,\epsilon}\|_{L^{2^*}(\Omega)} + \|u_{\theta,\epsilon}\|_{L^m(\Omega)} \leq K$ *(independent of* θ, ϵ*). Then there exists* $\theta_K > 0$ *such that for any* $\theta \in (0, \theta_K)$

$$|u_{\theta,\epsilon}(x)| \leq C|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} \quad \text{for all } x \in \Omega\backslash\{0\}, \tag{1.4}$$

where the constant $C = C(K)$ *does not depend on* θ, ϵ.

Especially, each solution $u_{\theta,0}$ *of* (1.3) *with* $\epsilon = 0$ *satisfies*

$$|u_{\theta,0}(x)| \leq C|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} \quad \text{for all } x \in \Omega\backslash\{0\}. \tag{1.5}$$

Furthermore, if $u_{\theta,0}$ *is positive, then there exists* $\rho_0 > 0$ *satisfying* $\overline{B_{\rho_0}(0)} \subset \Omega$, *and* $c_0 > 0$ *such that*

$$u_{\theta,0}(x) \geq c_0|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}} \quad \text{for all } x \in B_{\rho_0}(0)\backslash\{0\}. \tag{1.6}$$

Remark 1.2: We do not know if $\theta > 0$ is large, whether the constant C in (1.4) depends on ϵ, because our method fails. In addition, it follows from Theorem 1.1 that if $\epsilon = 0$, it is impossible to obtain the uniform L^∞-bound for the nontrivial solutions of problem (1.3) as in [11].

THEOREM 1.3: *Assume that* $N \geq 7$, $m \in (2, 2^*)$, $\mu \in (0, \bar{\mu})$ *and* $\lambda > 0$. *Then for any given positive integer* L, *there exists* $\theta_L > 0$ *such that for every* $\theta \in (0, \theta_L)$, *problem* (1.1) *admits at least* L *different solutions.*

In this paper, we consider another problem with concave and convex nonlinearities:

$$\begin{cases} -\Delta u = t|u|^{q-1}u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $q \in (0, 1)$, $t > 0$.

In recent years, people also have paid much attention to problem (1.7) and obtained many important and interesting results. A. Ambrosetti, H. Brezis and G. Cerami [1] proposed one open problem on (1.7): whether problem (1.7) has infinitely many solutions with positive energy for $t > 0$ small enough. Using the similar approaches in [11], we establish the existence of infinitely many solutions for problem (1.7), which gives a partial positive answer to this open problem proposed in [1]. More presisely, THEOREM 1.4: *Let $t > 0$ be fixed, $N \geq 7$, $4/(N-2) < q < 1$. Then problem* (1.7) *admits infinitely many solutions with positive energy.*

Remark 1.5: In Theorems 1.3, 1.4, we restrict the dimension $N \geq 7$, which seems reasonable. Also in the case $N \geq 7$, S. Solimini [19] proved that if $t \in (0, \lambda_1)$, problem (1.7) with $q = 1$ admits infinitely many radial solutions. In the case $N = 4, 5, 6$, it has been proved by F. V. Atkinson, H. Brezis and L. A. Peletier [2] that problem (1.7) with $q = 1$ does not have a radial solution which changes sign. Therefore, for the dimension $N = 4, 5, 6$, Theorems 1.3, 1.4 seem to be false unless the solutions found are identically zero or do not change sign.

The paper is organized as follows. In Section 2, using Moser type iteration and taking some ideas from [14], we characterize the asymptotic behavior of solutions of (1.3) at the origin (cf., Theorem 1.1). In Section 3, we deal with (1.3) with $\epsilon > 0$, and establish a strong convergence of solutions for (1.3) in $H_0^1(\Omega)$ through concentration estimates (cf., Proposition 3.1), where $p \in [2, 2^*]$, $m \in (2, 2^*)$. Section 4 devotes to the proof of Theorem 1.3. We first prove the existence of infinitely many solutions of

$$(1.8) \qquad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2+\epsilon} = \lambda u + |u|^{m-2}u + \theta |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\epsilon, \theta > 0$. Then if $\theta > 0$ is suitably small, we obtain the desired results by taking $\epsilon \to 0$. In Section 5, we deal with problem (1.7), and also establish the existence of infinitely many solutions (cf., Theorem 1.4).

Throughout this paper, we denote the norm of $H_0^1(\Omega)$ by $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{1/2}$; the norm of $L^l(\Omega)(1 \leq l < \infty)$ by $\|u\|_{L^l(\Omega)} = (\int_\Omega |u|^l dx)^{\frac{1}{l}}$, the norm of $L^\infty(\Omega)$ by $\|u\|_{L^\infty(\Omega)} = ess\sup_\Omega |u(x)|$ and positive constants (possibly different) by $C, C_1, C_2, \ldots$.

2. Asymptotic behavior of solutions of (1.3)

In this section, we prove Theorem 1.1 by referring to some of the techniques already developed by Felli–Schneider in [14].

Proof of Theorem 1.1. From the proofs of Theorems 1.1, 1.2 in [14], we can deduce that if $\epsilon = 0$, $|x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}} u_{\theta,0}(x)$ is Hölder continuous in Ω, and then has a positive lower bound on $B_{\rho_0}(0)$ if $u_{\theta,0}$ is positive, which imply that (1.5) (1.6) hold, where $\overline{B_{\rho_0}(0)} \subset \Omega$.

Now we prove that (1.4) holds for $\epsilon \geq 0$. We need to reveal that the constant C in (1.4) is independent of θ, ϵ. Set $v(x) = |x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} u_{\theta,\epsilon}(x)$. By using Hardy inequality

$$\int_\Omega \frac{|u|^2}{|x|^2} dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\nabla u|^2 dx \quad \text{for all } u \in H_0^1(\Omega),$$

we infer that $v \in H_0^1(\Omega, |x|^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} dx)$ and satisfies

$$\begin{aligned} (2.1) \qquad & -\operatorname{div}\left(|x|^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} \nabla v\right) \\ & = -\mu\epsilon \frac{v}{(|x|^2+\epsilon)|x|^{2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}+1)}} + \lambda |x|^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} v \\ & \quad + |x|^{-m(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} |v|^{m-2} v + \theta |x|^{-p(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} |v|^{p-2} v \end{aligned}$$

Choose

$$\varphi = v v_l^{2(s-1)} \in H_0^1(\Omega, |x|^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} dx), \quad s, l > 1, \quad v_l = \min\{|v|, l\},$$

and note that

$$\mu\epsilon \int_\Omega \frac{v\varphi}{(|x|^2+\epsilon)|x|^{2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}+1)}} = \mu\epsilon \int_\Omega \frac{|v|^2 v_l^{2(s-1)}}{(|x|^2+\epsilon)|x|^{2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}+1)}} \geq 0.$$

Multiplying both sides (2.1) by φ, we conclude

(2.2)

$$\begin{aligned} & \int_\Omega |x|^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} \left(v_l^{2(s-1)} |\nabla v|^2 + 2(s-1) v_l^{2(s-1)} |\nabla v_l|^2\right) dx \\ & \quad \leq \lambda \int_\Omega |x|^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} |v|^2 v_l^{2(s-1)} dx + \int_\Omega |x|^{-m(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} |v|^m v_l^{2(s-1)} dx \\ & \qquad + \theta \int_\Omega |x|^{-p(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})} |v|^p v_l^{2(s-1)} dx \end{aligned}$$

Now we recall Caffarelli–Kohn–Nirenberg's inequality (see [7, 10, 20]):

(2.3)

$$\left(\int_\Omega |x|^{-bt} |w|^t dx\right)^{2/t} \leq C_{a,b} \int_\Omega |x|^{-2a} |\nabla w|^2 dx \quad \text{for all } w \in H_0^1(\Omega, |x|^{-2a} dx),$$

where $-\infty < a < (N-2)/2$, $a \leq b \leq a+1$, $t = \frac{2N}{N-2+2(b-a)}$ and $C_{a,b}$ is a positive constant depending on a, b.

In the sequel, we take $a = b = \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu} < (N-2)/2$ in (2.3), then $t = 2^*$. Choosing $w = vv_l^{s-1}$ in (2.3), together with (2.2), we derive

$$
\begin{aligned}
&\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^{2^*}dx\right)^{2/2^*}\\
&\qquad\le C_{a,a}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|\nabla(vv_l^{s-1})|^2dx\\
&\qquad\le C\lambda s\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2(s-1)}dx\\
&\qquad\quad+Cs\int_\Omega |x|^{-m(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^m v_l^{2(s-1)}dx\\
&\qquad\quad+C\theta s\int_\Omega |x|^{-p(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^p v_l^{2(s-1)}dx.
\end{aligned}
\tag{2.4}
$$

Since $2 < m < 2^*$, we can choose $t_0 > N/2$ such that $(m-2)t_0 \le m$. Using the assumption $\|u_{\theta,\epsilon}\|_{L^m(\Omega)} \le K$ (independent of θ, ϵ), and noting that $2 < 2t_0/(t_0-1) < 2^*$ for $t_0 > N/2$. We deduce that for any $\delta > 0$,

$$
\begin{aligned}
&\int_\Omega |x|^{-m(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^m v_l^{2(s-1)}dx\\
&\qquad\le \|u_{\theta,\epsilon}\|^{m-2}_{L^{(m-2)t_0}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}vv_l^{s-1}\|^2_{L^{\frac{2t_0}{t_0-1}}(\Omega)}\\
&\qquad\le C\delta^2\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^{2^*}dx\right)^{2/2^*}\\
&\qquad\quad+C\delta^{\frac{-2N}{2t_0-N}}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^2dx.
\end{aligned}
\tag{2.5}
$$

Inserting (2.5) into (2.4), we get

$$
\begin{aligned}
&\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^{2^*}dx\right)^{2/2^*}\\
&\qquad\le C\lambda s\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2v_l^{2(s-1)}dx\\
&\qquad\quad+Cs\delta^2\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^{2^*}dx\right)^{2/2^*}\\
&\qquad\quad+Cs\delta^{-\frac{2N}{2t_0-N}}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^2dx\\
&\qquad\quad+C\theta s\int_\Omega |x|^{-p(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^p v_l^{2(s-1)}dx.
\end{aligned}
\tag{2.6}
$$

Taking $\delta = 1/\sqrt{2Cs}$ in (2.6), we conclude that

$$\begin{aligned}\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^{2^*}dx\right)^{2/2^*} \\ \le Cs^{\alpha_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2 v_l^{2(s-1)}dx \\ + C\theta s\int_\Omega |x|^{-p(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^p v_l^{2(s-1)}dx,\end{aligned} \tag{2.7}$$

where $\alpha_0 = 2t_0/(2t_0-N) > 0$, C is independent of θ, ϵ.

Choose $t_1 = N/2$, then $(p-2)t_1 \le 2^*$ for $p \in [2, 2^*]$. Noting that $2t_1/(t_1-1) = 2^*$, we deduce that

$$\begin{aligned}&\int_\Omega |x|^{-p(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^p v_l^{2(s-1)}dx \\ &\le \|u_{\theta,\epsilon}\|^{p-2}_{L^{(p-2)t_1}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}vv_l^{s-1}\|^2_{L^{\frac{2t_1}{t_1-1}}(\Omega)} \\ &\le C\|u_{\theta,\epsilon}\|^{p-2}_{L^{2^*}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}vv_l^{s-1}\|^2_{L^{2^*}(\Omega)}.\end{aligned} \tag{2.8}$$

Inserting (2.8) into (2.7), we obtain that

$$\begin{aligned}\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{s-1}\right|^{2^*}dx\right)^{2/2^*} \\ \le Cs^{\alpha_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2 v_l^{2(s-1)}dx \\ + C\theta s\|u_{\theta,\epsilon}\|^{p-2}_{L^{2^*}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}vv_l^{s-1}\|^2_{L^{2^*}(\Omega)}.\end{aligned} \tag{2.9}$$

Choosing $s = m/2 > 1$ in (2.9). Then we infer

$$\begin{aligned}\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{\frac{m}{2}-1}\right|^{2^*}dx\right)^{2/2^*} \\ \le C\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2 v_l^{2(\frac{m}{2}-1)}dx \\ + C\theta\|u_{\theta,\epsilon}\|^{p-2}_{L^{2^*}(\Omega)}\||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}vv_l^{\frac{m}{2}-1}\|^2_{L^{2^*}(\Omega)}.\end{aligned} \tag{2.10}$$

By the assumption $\theta^{1/2^*}\|u_{\theta,\epsilon}\|_{L^{2^*}(\Omega)} \le K$ (independent of θ, ϵ), then there exists $\theta_K > 0$ such that $C\theta\|u_{\theta,\epsilon}\|^{p-2}_{L^{2^*}(\Omega)} \le 1/2$ for every $\theta \in (0, \theta_K)$. From

(2.10), we conclude that

(2.11)

$$\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|vv_l^{\frac{m}{2}-1}\right|^{2^*}dx\right)^{2/2^*} \le C\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2 v_l^{2(\frac{m}{2}-1)}dx \le C(diam\Omega)^{(m-2)(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\|u_{\theta,\epsilon}\|^m_{L^m(\Omega)} \le C(N,\mu,\lambda,m),$$

where C depends on K, but not θ, ϵ.

Therefore, from (2.11), we obtain that by taking the limit $l \to \infty$

$$\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{\frac{m2^*}{2}}dx \le C. \tag{2.12}$$

Now we claim that there exists $\beta \in (1, m/2)$ such that

$$\int_\Omega |u_{\theta,\epsilon}|^{\beta 2^*}dx \le C \quad \text{where the constant } C \text{ is independent of } \theta,\ \epsilon. \tag{2.13}$$

In fact, from (2.12) and using Hölder inequality, we deduce for any $\beta \in (1, m/2)$

(2.14)

$$\begin{aligned}\int_\Omega |u_{\theta,\epsilon}|^{\beta 2^*}dx &= \int_\Omega |x|^{-\beta 2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{\beta 2^*}dx \\ &\le \left(\int_\Omega |x|^{-\beta 2^*(1-2/m)(1-2\beta/m)^{-1}(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx\right)^{1-2\beta/m} \\ &\quad\times\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^{\frac{m2^*}{2}}dx\right)^{2\beta/m} \\ &\le C\left(\int_0^{R_0} t^{N-1-\beta 2^*(1-2/m)(1-2\beta/m)^{-1}(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dt\right)^{1-2\beta/m},\end{aligned}$$

where $R_0 = \text{diam }\Omega$, and C is independent of θ, ϵ.

Note that as $\beta \to 1$,

$$N - \beta 2^*(1-2/m)(1-(2\beta)/m)^{-1}(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}) \to N - 2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}) = 2^*\sqrt{\bar\mu-\mu} > 0.$$

We infer that there exists $\beta \in (1, m/2)$ such that

$$N - 1 - \beta 2^*(1-2/m)(1-(2\beta)/m)^{-1}(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}) > -1,$$

and then from (2.14), we get (2.13).

Set $t_2 = \beta 2^*/(2^*-2)$, then $t_2 > N/2$ and $2t_2/(t_2-1) \in (2, 2^*)$. We conclude that for any $\delta > 0$,

$$\begin{aligned}
&\int_\Omega |x|^{-p(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^p v_l^{2(s-1)}dx \\
&\qquad \le \|u_{\theta,\epsilon}\|^{p-2}_{L^{(p-2)t_2}(\Omega)} \||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} v v_l^{s-1}\|^2_{L^{\frac{2t_2}{t_2-1}}(\Omega)} \\
&\qquad \le C\|u_{\theta,\epsilon}\|^{p-2}_{L^{\beta 2^*}(\Omega)} \times \big(\delta \||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} v v_l^{s-1}\|_{L^{2^*}(\Omega)} \\
&\qquad\quad + C(N,t_2)\delta^{-\frac{N}{2t_2-N}} \||x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} v v_l^{s-1}\|_{L^2(\Omega)}\big)^2.
\end{aligned} \tag{2.15}$$

Inserting (2.15) into (2.7), we infer from (2.13)

$$\begin{aligned}
&\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|v v_l^{s-1}\right|^{2^*}dx\right)^{2/2^*} \\
&\qquad \le Cs\delta^2 \left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|v v_l^{s-1}\right|^{2^*}dx\right)^{2/2^*} \\
&\qquad\quad + Cs^{\alpha_0}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2 v_l^{2(s-1)}dx \\
&\qquad\quad + Cs\delta^{-2N/(2t_2-N)}\int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|v v_l^{s-1}\right|^2 dx.
\end{aligned} \tag{2.16}$$

Taking $\delta = 1/\sqrt{2Cs}$ in (2.16), we conclude that

$$\left(\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\left|v v_l^{s-1}\right|^{2^*}dx\right)^{\frac{2}{2^*}} \le Cs^\alpha \int_\Omega |x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}|v|^2 v_l^{2(s-1)}dx, \tag{2.17}$$

where $\alpha = \max\{\alpha_0, 2t_2/(2t_2-N)\} > 0$, and C is independent of $\theta,\ \epsilon$.

Define the sequence $s_j = \big(2^*/2\big)^{j+1}$ $j = 0,1,2,\ldots$, and take $s = s_j$ in (2.17). Through a standard Moser type iteration procedure, we conclude from (2.13) and (2.17)

$$\|v_l\|_{L^{2s_{j+1}}(\Omega)} \le C|\Omega|^{1/2^*(1-1/\beta)}\left(\int_\Omega |u_{\theta,\epsilon}|^{\beta 2^*}dx\right)^{1/(\beta 2^*)} \le C,$$

where $C = C(K)$ is independent of θ, ϵ.

Note that $s_{j+1} \to \infty$ as $j \to \infty$. So let $j \to \infty$ in the above inequality, we infer that $\|v_l\|_{L^\infty(\Omega)} \le C$, and (1.4) can be obtained by taking $l \to +\infty$. ∎

3. H^1-strong convergence of solutions for (1.3)

In the next parts, we suppose $\epsilon > 0$. Following the arguments of Theorem 1.1 in [11], we establish strong convergence in $H_0^1(\Omega)$ on solutions for (1.3), that is,

Proposition 3.1: *Assume that $N \geq 7$, $\mu \geq 0$, $\lambda > 0$. Then for any sequence $\{u_p\}$, which are solutions of (1.3) satisfying $\|u_p\|_{H_0^1(\Omega)} \leq C$ for some constant C independent of $p \in (2, 2^*)$, there exists a subsequence of $\{u_p\}$, which converges strongly in $H_0^1(\Omega)$ as $p \to 2^*$.*

Before giving the proof of Proposition 3.1, we introduce some notation and terminology, which can be found in [11].

Let u be a solution of problem (1.3). Set $v = |u|$ (extended by zero out of Ω), then $v \in H^1(R^N)$ satisfies

$$-\Delta v \leq bv^{2^*-1} + A, \tag{3.1}$$

where $A = A(\lambda, \mu, \epsilon) > 0$ is independent of u, $b > 1$. In the next, we normalize b and always take $b = 1$ in (3.1).

Definition 3.2: $\{u_n\} \subset H_0^1(\Omega)$ is said to be a controlled sequence if each u_n is a solution to problem (3.1); a balanced sequence if for some $p \in (2, 2^*)$, u_n solves problem (1.3) with $\epsilon > 0$.

Let S be the best Sobolev constant defined by

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}},$$

which is achieved if and only if $\Omega = R^N$ by

$$U(x) = \frac{(N(N-2))^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}.$$

The function U, called an instanton, satisfies

$$-\Delta U = U^{2^*-1} \quad \text{in } R^N.$$

Moreover,

$$\int_{R^N} |\nabla U|^2 dx = \int_{R^N} U^{2^*} dx = S^{N/2}.$$

The proof of the following result can be found in [21]:

Let $\{u_n\} \subset H_0^1(\Omega)$ be a Palais–Smale sequence of $I_{\theta,\epsilon}$, which is defined in (4.2). Then, up to a subsequence, there exist k sequences of mutually diverging scaling σ_n^i with respective concentration points x_n^i such that as $n \to \infty$

$$(3.2) \quad u_n - \sum_{i=1}^{k} (\sigma_n^i)^{(N-2)/2} U(\sigma_n^i (x - x_n^i)) - u_\infty \longrightarrow 0 \quad \text{strongly in } H_0^1(\Omega),$$

where u_∞ is a weak solution of problem (1.8).

We call $\{u_n\}$ is a concentrating sequence if the limit in (3.2) holds in H^1-strong topology.

The following result is from Lemma 6.2 in [11], which allows us to extract a concentrating subsequence from a noncompact bounded balanced sequence.

Lemma 3.3: *Let $\{u_n\} \subset H_0^1(\Omega)$ be a noncompact bounded balanced sequence. Then we can always extract a concentrating subsequence from $\{u_n\}$.*

Choose a constant $\overline{C} > 0$ such that $\mathcal{A}_n^1 = B_{(\overline{C}+5)\sigma_n^{-1/2}}(x_n) \backslash B_{\overline{C}\sigma_n^{-1/2}}(x_n)$ does not contain any concentration point for sufficiently large n.

Define the thinner subset $\mathcal{A}_n^2 = B_{(\overline{C}+4)\sigma_n^{-1/2}}(x_n) \backslash B_{(\overline{C}+1)\sigma_n^{-1/2}}(x_n)$. Then the following results on the controlled concentrating sequences hold (see [11]):

Lemma 3.4: *Let $\{u_n\}$ be a controlled concentrating sequence. Then there exists a $t_n \in [\overline{C}+2, \overline{C}+3]$ such that*

$$u_n(x) \le C, \quad \text{for all } x \in \mathcal{A}_n^2 \quad \text{and} \quad \int_{\partial B_{t_n \sigma_n^{-1/2}}(x_n)} |\nabla u_n|^2 d\sigma \le C \sigma_n^{-(N-3)2}.$$

Proof of Proposition 3.1. We prove Proposition 3.1 by contradiction. Assume the bounded balanced sequence $\{u_p\}$ is not compact. Then by Lemma 3.3, we can choose a concentrating subsequence of $\{u_p\}$, denoted by $\{u_n\}$ with $p = p_n < 2^*$, $p_n \to 2^*$. Thus, to prove the strong convergence in $H_0^1(\Omega)$, we just need to show the bubbles $(\sigma_n^i)^{(N-2)/2} U(\sigma_n^i(x - x_n^i))$ $(1 \le i \le k)$ in (3.2) will not appear in the decomposition of u_n. Assume $\|u_n\|_{H_0^1(\Omega)} \le M$, then from (3.2), we infer that $k < \infty$. For simplicity, we take $\sigma_n = \sigma_n^i$, $x_n = x_n^i$. Since the proof is similar to that of Lemma 6.1 in [11], here we only give a sketch of it. After a detailed calculation, we have the local Pohozaev identity for $\{u_n\}$ on $B_n = B_{t_n \sigma_n^{-1/2}}(x_n) \cap \Omega$

$$\begin{aligned}
&\theta\Big(\frac{N}{p_n}-\frac{N-2}{2}\Big)\int_{B_n}|u_n|^{p_n}dx+\Big(\frac{N}{m}-\frac{N-2}{2}\Big)\int_{B_n}|u_n|^m dx\\
&\quad+\epsilon\mu\int_{B_n}\frac{|u_n|^2}{(|x|^2+\epsilon)^2}dx+\lambda\int_{B_n}|u_n|^2dx+\mu\int_{B_n}\frac{x\cdot x_0|u_n|^2}{(|x|^2+\epsilon)^2}dx\\
&\quad=\frac{\theta}{p_n}\int_{\partial B_n}|u_n|^{p_n}(x-x_0)\cdot\nu d\sigma+\frac{1}{m}\int_{\partial B_n}|u_n|^m(x-x_0)\cdot\nu d\sigma\\
&\qquad+\frac{1}{2}\int_{\partial B_n}\Big(\lambda+\frac{\mu}{|x|^2+\epsilon}\Big)|u_n|^2(x-x_0)\cdot\nu d\sigma\\
&\qquad+\int_{\partial B_n}(\nabla u_n\cdot(x-x_0))(\nabla u_n\cdot\nu)d\sigma-\frac{1}{2}\int_{\partial B_n}|\nabla u_n|^2(x-x_0)\cdot\nu d\sigma\\
&\qquad+\frac{N}{2^*}\int_{\partial B_n}\nabla u_n\cdot\nu u_n d\sigma,
\end{aligned}\tag{3.3}$$

where ν is the outward normal to ∂B_n.

Set $\partial B_n=\partial_i B_n\cup\partial_e B_n$, where $\partial_i B_n=\partial B_n\cap\Omega$, $\partial_e B_n=\partial\Omega\cap\overline{B_n}$. As in [11], if $\partial_e B_n=\emptyset$, we take x_0 in (3.3) equal to the concentration point x_n; if $\partial_e B_n\neq\emptyset$, we take x_0 out of Ω such that

$$d(x_0,x_n)\le 2t_n\sigma_n^{-\frac{1}{2}}\quad\text{and}\quad\forall x\in\partial_e B_n,\quad \nu\cdot(x-x_0)<0.\tag{3.4}$$

Hence,

$$\begin{aligned}
&\lambda\int_{B_n}|u_n|^2dx+\mu\int_{B_n}\frac{x\cdot x_0|u_n|^2}{(|x|^2+\epsilon)^2}dx\\
&\quad=\lambda\int_{B_n}|u_n|^2dx+\mu\int_{B_n}\frac{x\cdot(x_0-x)|u_n|^2}{(|x|^2+\epsilon)^2}dx+\mu\int_{B_n}\frac{|x|^2|u_n|^2}{(|x|^2+\epsilon)^2}dx\\
&\quad\ge\lambda\int_{B_n}|u_n|^2dx-\mu\int_{B_n}\frac{|x||x_0-x||u_n|^2}{(|x|^2+\epsilon)^2}dx+\mu\int_{B_n}\frac{|x|^2|u_n|^2}{(|x|^2+\epsilon)^2}dx\\
&\quad\ge\big(\lambda-C(\mu,\epsilon)\sigma_n^{-1}\big)\int_{B_n}|u_n|^2dx\\
&\quad\ge C(\mu,\lambda,\epsilon)\int_{B_n}|u_n|^2dx\quad\text{for large}\quad n.
\end{aligned}\tag{3.5}$$

Let $B_n'=B_{\sigma_n^{-1}}(x_n)$ and $u_n=u_n^0+u_n^1+u_n^2$, where

$$u_n^1=u_\infty,\quad u_n^2=\sum_{i=1}^k(\sigma_n^i)^{\frac{N-2}{2}}U(\sigma_n^i(x-x_n^i)),\quad u_n^0=u_n-u_n^1-u_n^2.$$

Then we deduce that for n large enough, $B_n' \subset B_n \cap \Omega$ and

$$\int_{B_n\cap\Omega} |u_n|^2 dx \geq \int_{B_n'} |u_n|^2 dx \geq \frac{1}{2}\int_{B_n'} |u_n^2|^2 dx - 2\int_{B_n'} |u_n^1|^2 dx - 2\int_{B_n'} |u_n^0|^2 dx. \tag{3.6}$$

After a direct calculation, we have

$$\int_{B_n'} |u_n^2|^2 dx \geq C\sigma_n^{-2}, \quad \int_{B_n'} |u_n^1|^2 dx \leq C\sigma_n^{-N}, \quad \int_{B_n'} |u_n^0|^2 dx \leq C\|u_n^0\|^2_{L^{2^*}(\Omega)}\sigma_n^{-2}. \tag{3.7}$$

Note that $\|u_n^0\|_{L^{2^*}(\Omega)} \to 0$ as $n \to \infty$. Inserting (3.7) into (3.6), we get for n large enough

$$\int_{B_n\cap\Omega} |u_n|^2 dx \geq C\sigma_n^{-2}. \tag{3.8}$$

By the choice of x_0, as in [11], we only need to consider the right hand side of (3.3) on $\partial_i B_n$. Using Lemma 3.4, we get

$$\begin{aligned}
&\frac{\theta}{p_n}\int_{\partial_i B_n} |u_n|^{p_n}(x-x_0)\cdot\nu d\sigma + \frac{1}{m}\int_{\partial_i B_n} |u_n|^m (x-x_0)\cdot\nu d\sigma \\
&\quad + \frac{1}{2}\int_{\partial_i B_n} \big(\lambda + \frac{\mu}{|x|^2+\epsilon}\big)|u_n|^2(x-x_0)\cdot\nu d\sigma \\
&\quad + \int_{\partial_i B_n} (\nabla u_n\cdot(x-x_0))(\nabla u_n\cdot\nu) d\sigma \\
&\quad - \frac{1}{2}\int_{\partial_i B_n} |\nabla u_n|^2 (x-x_0)\cdot\nu d\sigma + \frac{N}{2^*}\int_{\partial_i B_n} \nabla u_n\cdot\nu u_n d\sigma \\
&\qquad \leq C\int_{\partial_i B_n} |(x-x_0)\cdot\nu| d\sigma + \int_{\partial_i B_n} |\nabla u_n|^2|x-x_0| d\sigma \\
&\qquad\quad + \Big(\int_{\partial_i B_n} |\nabla u_n|^2 d\sigma\Big)^{1/2}\Big(\int_{\partial_i B_n} |u_n|^2 d\sigma\Big)^{1/2} \\
&\qquad \leq C\sigma_n^{-\frac{N-2}{2}}.
\end{aligned} \tag{3.9}$$

Note that $\epsilon\mu\int_{B_n}\frac{|u_n|^2}{(|x|^2+\epsilon)^2}dx > 0$, $\theta(N/p_n - (N-2)/2)\int_{B_n}|u_n|^{p_n}dx > 0$ for $p_n \in (2, 2^*)$ and $(N/m - (N-2)/2)\int_{B_n}|u_n|^m dx > 0$ for $m \in (2, 2^*)$. Inserting (3.8), (3.9) into (3.3), we obtain

$$\sigma_n^{-2} \leq C(\epsilon)\sigma_n^{-\frac{N-2}{2}},$$

which is a contradiction for n large enough due to $N \geq 7$. Thus, all of the bubbles in (3.2) must vanish. ∎

4. Existence of many solutions for (1.1)

In this section, we first introduce some notation (see [21]) and preliminary lemmas.

Denote the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and the corresponding eigenfunctions by $e_1(x), e_2(x), \ldots$. Then, $\{e_i(x)\}_{i=1}^{\infty}$ consist of an orthogonal basis in $H_0^1(\Omega)$.

It is well-known that the nontrivial solutions of problems (1.3), (1.8) are the corresponding nonzero critical points of the following energy functionals defined on $H_0^1(\Omega)$ respectively:

(4.1)
$$I_{\theta,\epsilon}^{(p)}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2+\epsilon} - \lambda|u|^2\right)dx - \frac{1}{m}\int_\Omega |u|^m dx - \frac{\theta}{p}\int_\Omega |u|^p dx,$$

and

(4.2)
$$I_{\theta,\epsilon}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2+\epsilon} - \lambda|u|^2\right)dx - \frac{1}{m}\int_\Omega |u|^m dx - \frac{\theta}{2^*}\int_\Omega |u|^{2^*} dx.$$

Set $Y_k := \bigoplus_{j=1}^k e_j$, $Z_k := \overline{\bigoplus_{j=k}^\infty e_j}$ and

$$B_k := \{u \in Y_k \colon \|u\|_{H_0^1(\Omega)} \leq \rho_k\}, \quad N_k := \{u \in Z_k \colon \|u\|_{H_0^1(\Omega)} = r_k\}$$

where $\rho_k > r_k > 0$.

Define

$$\Gamma_k := \{\gamma \in C(B_k, H_0^1(\Omega)) \colon \gamma|_{\partial B_k} = \mathrm{id}\}, \quad c_{k,\theta} := \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} I_{\theta,0}(\gamma(u))$$

$$c_{k,\theta,\epsilon} := \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} I_{\theta,\epsilon}(\gamma(u)), \qquad c_{k,\theta,\epsilon}^n := \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} I_{\theta,\epsilon}^{(p_n)}(\gamma(u)),$$

$$b_{k,\theta,\epsilon} := \inf_{u\in N_k} I_{\theta,\epsilon}(u), \qquad b_{k,\theta,\epsilon}^n := \inf_{u\in N_k} I_{\theta,\epsilon}^{(p_n)}(u),$$

where $p_n \in (2, 2^*)$ and $p_n \to 2^*$ as $n \to \infty$.

Lemma 4.1: *For any $\epsilon, \theta > 0$ and positive integer k,* $\lim_{n\to\infty} c_{k,\theta,\epsilon}^n = c_{k,\theta,\epsilon}$.

Proof. For any $v \in H_0^1(\Omega)$,

$$I_{\theta,\epsilon}(v) = I_{\theta,\epsilon}^{(p_n)}(v) + \frac{\theta}{p_n}\int_\Omega |v|^{p_n}dx - \frac{\theta}{2^*}\int_\Omega |v|^{2^*}dx. \tag{4.3}$$

Since for $s \geq 0$, the function $h(s) = \frac{\theta}{p_n}s^{p_n} - \frac{\theta}{2^*}s^{2^*}$ attains its maximum value at $s = 1$, we have $h(s) \leq \theta/p_n - \theta/2^*$ for all $s \geq 0$. Therefore, for every $v \in H_0^1(\Omega)$,

$$I_{\theta,\epsilon}(v) \leq I_{\theta,\epsilon}^{(p_n)}(v) + \theta\big(1/p_n - 1/2^*\big)|\Omega|.$$

So, for any $\epsilon > 0$ and positive integer k,

$$c_{k,\theta,\epsilon} = \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}(\gamma(u)) \leq \underline{\lim}_{n\to\infty}\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}^{(p_n)}(\gamma(u)) = \underline{\lim}_{n\to\infty} c_{k,\theta,\epsilon}^n, \tag{4.4}$$

Set

$$F^{(p_n)}(v) = \frac{\theta}{2^*}\int_\Omega |v|^{2^*}dx - \frac{\theta}{p_n}\int_\Omega |v|^{p_n}dx \quad v \in H_0^1(\Omega). \tag{4.5}$$

Note that id $\in \Gamma_k$, we deduce from (4.3), (4.5) that

$$\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}^{(p_n)}(\gamma(u)) \leq \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}(\gamma(u)) + \sup_{u\in B_k} F^{(p_n)}(u). \tag{4.6}$$

Since B_k is compact and the functionals $F^{(p_n)}$ are equicontinuous on B_k, we derive that $\lim_{n\to\infty}\sup_{u\in B_k} F^{(p_n)}(u) \to 0$. So from (4.6), we get

$$\begin{aligned} \overline{\lim}_{n\longrightarrow\infty} c_{k,\theta,\epsilon}^n &= \overline{\lim}_{n\longrightarrow\infty}\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}^{(p_n)}(\gamma(u)) \\ &\leq \overline{\lim}_{n\longrightarrow\infty}\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}(\gamma(u)) + \overline{\lim}_{n\longrightarrow\infty}\sup_{u\in B_k} F^{(p_n)}(u) \\ &= c_{k,\theta,\epsilon}. \end{aligned} \tag{4.7}$$

Therefore, from (4.4) and (4.7), we infer that $\lim_{n\to\infty} c_{k,\theta,\epsilon}^n = c_{k,\theta,\epsilon}$. ∎

LEMMA 4.2: $\lim_{k\to\infty} c_{k,\theta,\epsilon} = +\infty$ *for every* $\epsilon, \theta > 0$.

Proof. It follows from Lemma 4.1 that for every positive integer k, there exists $n_k > k$ such that for any $\epsilon, \theta > 0$

$$\left|c_{k,\theta,\epsilon}^{n_k} - c_{k,\theta,\epsilon}\right| < 1/k. \tag{4.8}$$

Let $\delta_0 \in (0, \lambda_1)$ be a fixed number. Define

$$\alpha_k := \inf_{u\in Z_k,\ \|u\|_{L^{p_{n_k}}(\Omega)}=1}\int_\Omega (|\nabla u|^2 - \delta_0|u|^2)dx.$$

We claim that, up to a subsequence, $\alpha_k \to +\infty$ as $k \to \infty$. In fact, since $p_{n_k} < 2^*$, we infer that α_k can be achieved by a function $v_k \in Z_k$, $\int_\Omega |v_k|^{p_{n_k}} dx = 1$, which satisfies

$$-\Delta v_k = \alpha_k |v_k|^{p_{n_k}-2} v_k + \delta_0 v_k.$$

If $\alpha_k \nrightarrow \infty$ as $k \to \infty$, then $\int_\Omega |\nabla v_k|^2 dx \leq C$ by the choice of δ_0. By using Theorem 1.1 in [11], we conclude that

$$\sup_k \|v_k\|_{L^\infty(\Omega)} \leq C. \tag{4.9}$$

Since $v_k \in Z_k$, up to a subsequence, we may assume that

$$v_k \rightharpoonup 0 \quad \text{weakly in} \quad H_0^1(\Omega); \quad v_k \to 0 \quad \text{a.e. on } \Omega.$$

By (4.9) and the dominated convergence theorem, we deduce that

$$\lim_{k\to\infty} \int_\Omega |v_k|^{p_{n_k}} dx = 0,$$

which is a contradiction due to $\int_\Omega |v_k|^{p_{n_k}} dx = 1$. Thus $\alpha_k \to \infty$ as $k \to \infty$. Note that $p_{n_k} \in (2, 2^*)$, $p_{n_k} \to 2^*$ as $k \to \infty$, we may assume $m < p_{n_k}$ for large k. Then by Young inequality, we have for any $u \in Z_k$

$$\begin{aligned} I_{\theta,\epsilon}^{(p_{n_k})}(u) &= \frac{1}{2}\int_\Omega \big(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2}\big)dx - \frac{1}{m}\int_\Omega |u|^m dx - \frac{\theta}{p_{n_k}}\int_\Omega |u|^{p_{n_k}} dx \\ &\quad - \frac{\lambda}{2}\int_\Omega |u|^2 dx + \frac{\mu}{2}\int_\Omega \big(\frac{|u|^2}{|x|^2} - \frac{|u|^2}{|x|^2+\epsilon}\big)dx \\ &\geq C_1\|u\|^2_{H_0^1(\Omega)} - C_2\alpha_k^{-\frac{p_{n_k}}{2}}\|u\|^{p_{n_k}}_{H_0^1(\Omega)} - C_3. \end{aligned}$$

Choosing $r_k = \big(\frac{2C_1\alpha_k^{p_{n_k}/2}}{C_2 p_{n_k}}\big)^{\frac{1}{p_{n_k}-2}}$, we obtain that if $u \in Z_k$ and $\|u\|_{H_0^1(\Omega)} = r_k$,

$$I_{\theta,\epsilon}^{(p_{n_k})}(u) \geq C_1\Big(1 - \frac{2}{p_{n_k}}\Big)\Big(\frac{2C_1\alpha_k^{p_{n_k}/2}}{C_2 p_{n_k}}\Big)^{\frac{2}{p_{n_k}-2}} - C_3. \tag{4.10}$$

Since we have proved that $\alpha_k \to \infty$ as $k \to \infty$, from (4.10), we infer that $b^{n_k}_{k,\theta,\epsilon} \to \infty$ as $k \to \infty$. It follows from Theorem 3.5 in [21] that $c^{n_k}_{k,\theta,\epsilon} \geq b^{n_k}_{k,\theta,\epsilon}$, and so from (4.8), we get that $\lim_{k\to\infty} c_{k,\theta,\epsilon} = \lim_{k\to\infty} c^{n_k}_{k,\theta,\epsilon} = +\infty$. ∎

Lemma 4.3: $\lim_{\epsilon\to 0} c_{k,\theta,\epsilon} = c_{k,\theta}$ *for any* $\theta > 0$ *and positive integer* k.

Proof. Observe that for any $v \in H_0^1(\Omega)$,

$$I_{\theta,\epsilon}(v) = I_{\theta,0}(v) + \frac{\mu}{2}\int_\Omega \Big(\frac{|u|^2}{|x|^2} - \frac{|u|^2}{|x|^2+\epsilon}\Big)dx. \tag{4.11}$$

Since id $\in \Gamma_k$, we have

(4.12)

$$\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,\epsilon}(\gamma(u)) \le \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} I_{\theta,0}(\gamma(u)) + \frac{\mu}{2}\sup_{u\in B_k}\int_\Omega \Big(\frac{|u|^2}{|x|^2} - \frac{|u|^2}{|x|^2+\epsilon}\Big)dx.$$

Set

$$G_\epsilon(u) = \int_\Omega \Big(\frac{|u|^2}{|x|^2} - \frac{|u|^2}{|x|^2+\epsilon}\Big)dx \quad u \in B_k.$$

By the dominated convergence theorem, we infer that for any $u \in B_k$, $\lim_{\epsilon\to 0} G_\epsilon(u) = 0$. Since B_k is compact and the functionals G_ϵ are equicontinuous on B_k, we conclude

$$\lim_{\epsilon\to 0}\sup_{u\in B_k} G_\epsilon(u) = 0.$$

Thus we obtain from (4.12) that

$$\overline{\lim_{\epsilon\to 0}} c_{k,\theta,\epsilon} \le c_{k,\theta}. \tag{4.13}$$

On the other hand, since $\int_\Omega \big(\frac{|u|^2}{|x|^2} - \frac{|u|^2}{|x|^2+\epsilon}\big)dx \ge 0$, from (4.11), we infer that $c_{k,\theta} \le \underline{\lim}_{\epsilon\to 0}\, c_{k,\theta,\epsilon}$. Together with (4.13), we conclude that

$$\lim_{\epsilon\to 0} c_{k,\theta,\epsilon} = c_{k,\theta}. \qquad \blacksquare$$

Proof of Theorem 1.3. It is not difficult to verify that the assumptions (A_1)–(A_4) of Theorem 3.6 in [21] are satisfied for problem (1.3) with $p = p_n \in (2, 2^*)$. So by Theorem 3.6 in [21], we conclude that $I_{\theta,\epsilon}^{(p_n)}$ has a sequence of critical points, denoted by $u_{k,\epsilon}^n$. Moreover, $c_{k,\theta,\epsilon}^n = I_{\theta,\epsilon}^{(p_n)}(u_{k,\epsilon}^n)$. By Lemma 4.1, we deduce that $\{u_{k,\epsilon}^n\}_{n=1}^\infty$ is bounded in $H_0^1(\Omega)$. Then, by Proposition 3.1, we can find a subsequence which converges to a solution $u_{k,\epsilon}$ of (1.8) at level $c_{k,\epsilon}$. Note that $c_{k,\theta} \le c_{k,0}$. From Lemma 4.3 and the following equality

$$c_{k,\theta,\epsilon} = \theta(1/2 - 1/2^*)\int_\Omega |u_{k,\epsilon}|^{2^*}dx + (1/2 - 1/m)\int_\Omega |u_{k,\epsilon}|^m dx,$$

which implies that

$$\theta^{\frac{1}{2^*}}\|u_{k,\epsilon}\|_{L^{2^*}(\Omega)} + \|u_{k,\epsilon}\|_{L^m(\Omega)} \le C(k), \quad \text{where } C(k) \text{ is independent of } \theta,\ \epsilon.$$

From Theorem 1.1, we infer that there exists $\theta_k > 0$ such that for every $\theta \in (0, \theta_k)$

$$|u_{k,\epsilon}(x)| \le C|x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} \quad \text{for all } x \in \Omega\backslash\{0\}, \tag{4.14}$$

where C is independent of ϵ.

Thus by Hardy inequality and from the following equality

$$c_{k,\theta,\epsilon} = \Big(\frac{1}{2}-\frac{1}{2^*}\Big)\int_\Omega \Big(|\nabla u_{k,\epsilon}|^2-\mu\frac{|u_{k,\epsilon}|^2}{|x|^2+\epsilon}-\lambda|u_{k,\epsilon}|^2\Big)dx-\Big(\frac{1}{m}-\frac{1}{2^*}\Big)\int_\Omega |u_{k,\epsilon}|^m dx,$$

we infer that

$$\int_\Omega |\nabla u_{k,\epsilon}|^2 dx \le C(k,\mu,\lambda), \tag{4.15}$$

and then by Sobolev inequality, we also have

$$\int_\Omega |u_{k,\epsilon}|^{2^*} dx \le C(k,\mu,\lambda). \tag{4.16}$$

From (4.15), up to a subsequence, we may assume that as $\epsilon \to 0$,

$$\begin{aligned} u_{k,\epsilon} &\rightharpoonup u_k \quad \text{weakly in } H_0^1(\Omega),\\ u_{k,\epsilon} &\rightharpoonup u_k \quad \text{weakly in } L^{2^*}(\Omega),\\ u_{k,\epsilon} &\to u_k \quad \text{a.e. on } \Omega. \end{aligned}$$

Then $u_k \in H_0^1(\Omega)$ is a weak solution of (1.1), and we deduce from (1.5) that

$$|u_k(x)| \le C|x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} \quad \text{for all } x \in \Omega\backslash\{0\}.$$

Together with (4.14), we infer that

$$|u_{k,\epsilon}(x)-u_k(x)|^{2^*} \le C|x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})} \quad \text{for all } x \in \Omega\backslash\{0\},$$

and

$$\frac{|u_{k,\epsilon}(x)-u_k(x)|^2}{|x|^2} \le C|x|^{-2-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})} \quad \text{for all } x \in \Omega\backslash\{0\},$$

where C is independent of ϵ.

After a direct calculation, we infer that

$$\int_\Omega |x|^{-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx \le C\int_0^{R_0} t^{N-1-2^*(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dt \le C,$$

and

$$\int_\Omega |x|^{-2-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dx \le C\int_0^{R_0} t^{N-1-2-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}dt \le C.$$

Thus by the dominated convergence theorem, we conclude that

(4.17)
$$\lim_{\epsilon\to\infty}\int_\Omega |u_{k,\epsilon}(x)-u_k(x)|^{2^*}dx=0 \quad\text{and}\quad \lim_{\epsilon\to\infty}\int_\Omega \frac{|u_{k,\epsilon}(x)-u_k(x)|^2}{|x|^2}=0.$$

Since $u_{k,\epsilon}$ and u_k are solutions of (1.8) and (1.1), respectively, we derive

$$\begin{aligned}&\int_\Omega |\nabla(u_{k,\epsilon}-u_k)|^2dx\\ &=\mu\int_\Omega\Big(\frac{u_{k,\epsilon}}{|x|^2+\epsilon}-\frac{u_k}{|x|^2}\Big)(u_{k,\epsilon}-u_k)dx+\lambda\int_\Omega|u_{k,\epsilon}-u_k|^2dx\\ &\quad+\int_\Omega(|u_{k,\epsilon}|^{m-2}u_{k,\epsilon}-|u_k|^{m-2}u_k)(u_{k,\epsilon}-u_k)dx\\ &\quad+\theta\int_\Omega(|u_{k,\epsilon}|^{2^*-2}u_{k,\epsilon}-|u_k|^{2^*-2}u_k)(u_{k,\epsilon}-u_k)dx.\end{aligned}\tag{4.18}$$

By Hardy's inequality and (4.15), we get

$$\begin{aligned}&\Big|\int_\Omega\Big(\frac{u_{k,\epsilon}}{|x|^2+\epsilon}-\frac{u_k}{|x|^2}\Big)(u_{k,\epsilon}-u_k)dx\Big|\\ &\le\int_\Omega\frac{|u_{k,\epsilon}(u_{k,\epsilon}-u_k)|}{|x|^2+\epsilon}dx+\int_\Omega\frac{|u_k(u_{k,\epsilon}-u_k)|}{|x|^2}dx\\ &\le\Big(\Big(\int_\Omega\frac{|u_{k,\epsilon}|^2}{|x|^2}dx\Big)^{1/2}+\Big(\int_\Omega\frac{|u_k|^2}{|x|^2}dx\Big)^{1/2}\Big)\Big(\int_\Omega\frac{|u_{k,\epsilon}-u_k|^2}{|x|^2}dx\Big)^{1/2}\\ &\le C\big(\|\nabla u_{k,\epsilon}\|_{L^2(\Omega)}+\|\nabla u_k\|_{L^2(\Omega)}\big)\Big(\int_\Omega\frac{|u_{k,\epsilon}-u_k|^2}{|x|^2}dx\Big)^{1/2}\\ &\le C\Big(\int_\Omega\frac{|u_{k,\epsilon}-u_k|^2}{|x|^2}dx\Big)^{1/2}.\end{aligned}\tag{4.19}$$

Hence from (4.17), (4.19), we infer that

$$\lim_{\epsilon\to0}\int_\Omega\Big(\frac{u_{k,\epsilon}}{|x|^2+\epsilon}-\frac{u_k}{|x|^2}\Big)(u_{k,\epsilon}-u_k)dx=0.\tag{4.20}$$

It is not difficult to verify from (4.17) that

$$\lim_{\epsilon\to0}\int_\Omega(|u_{k,\epsilon}|^{2^*-2}u_{k,\epsilon}-|u_k|^{2^*-2}u_k)(u_{k,\epsilon}-u_k)dx=0,\tag{4.21}$$

$$\begin{aligned}&\lim_{\epsilon\to0}\int_\Omega(|u_{k,\epsilon}|^{m-2}u_{k,\epsilon}-|u_k|^{m-2}u_k)(u_{k,\epsilon}-u_k)dx=0\quad\text{and}\\ &\qquad\lim_{\epsilon\to0}\int_\Omega|u_{k,\epsilon}-u_k|^2dx=0.\end{aligned}\tag{4.22}$$

Hence, from (4.18), (4.20)–(4.22), we deduce that $u_{k,\epsilon} \to u_k$ strongly in $H_0^1(\Omega)$ as $\epsilon \to 0$, and then u_k is a critical point of $I_{\theta,0}$ at the level $c_{k,\theta}$ for any $\theta \in (0, \theta_k)$. On the other hand, for every k, from Lemmas 4.2, 4.3, we infer that there exists $l > k$ such that $c_{k,\theta} \neq c_{l,\theta}$ for any $\theta > 0$. Hence, for any given positive integer L, there exist $k_1 < k_2 < \cdots < k_L$ and $\theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_L}$ (see Theorem 1.1) such that for any $\theta \in (0, \theta_L)$, $c_{k_i,\theta} \neq c_{k_j,\theta}$ and then $u_{k_i,\theta} \neq u_{k_j,\theta}$ in Ω for any $i \neq j$, $i, j = 1, 2, \ldots, L$, where $\theta_L = \min\{\theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_L}\}$. ∎

5. Existence of infinitely many solutions for (1.7)

In this section, we first establish a strong convergence in $H_0^1(\Omega)$ on solutions of

$$\begin{cases} -\Delta u = |u|^{p-2}u + t|u|^{q-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $p \in (2, 2^*)$, $q \in (0, 1)$ and $t > 0$. That is,

Proposition 5.1: *Assume that $N \geq 7$, $4/(N-2) < q < 1$, $t > 0$. Then any sequence $\{u_p\}$ of solutions of (5.1) with p varying in $(2, 2^*)$ satisfying $\|u_p\|_{H_0^1(\Omega)} \leq C$ for some constant C independent of p, has a subsequence which converges strongly in $H_0^1(\Omega)$ as $p \to 2^*$.*

Before giving the proof of Proposition 5.1, we also introduce some notation and terminology, which can be found in [11].

Let u be a solution of problem (5.1). Set $v = |u|$ (extended by zero out of Ω), then v satisfies

$$-\Delta v \leq bv^{2^*-1} + A \quad v \in H^1(R^N), \tag{5.2}$$

where $b > 1$ is a coefficient constant, $A = A(t, q)$ is a positive constant. By normalizing, in the next, we assume that $b = 1$ in (5.2) .

Definition 5.2: $\{u_n\} \subset H_0^1(\Omega)$ is said to be a controlled sequence if each u_n is a solution to problem (5.2); a balanced sequence if u_n solves problem (5.1) for some $p \in (2, 2^*)$.

Now we characterize the representation of Palais–Smale sequences corresponding to (1.7). Since its proof is similar to that of Theorem 8.13 in [21], we omit its details here.

Let $\{u_n\} \subset H_0^1(\Omega)$ be a Palais–Smale sequence of the functional corresponding to (1.7). Then, up to subsequence, there exists a positive integer k, k sequence of mutually diverging scaling σ_n^i with respective concentration points x_n^i such that as $n \longrightarrow \infty$

$$(5.3)\qquad u_n - \sum_{i=1}^{k} (\sigma_n^i)^{\frac{N-2}{2}} U(\sigma_n^i(x - x_n^i)) - u_\infty \longrightarrow 0 \quad \text{strongly in} \quad H_0^1(\Omega),$$

where u_∞ is a weak solution of problem (1.7).

We call $\{u_n\}$ is a concentrating sequence if the limit in (5.3) holds.

Lemma 5.3: *Let $\{u_n\}$ be a controlled concentrating sequence. Then there exists a $t_n \in [\overline{C}+2, \overline{C}+3]$ such that*

$$u_n(x) \leq C, \quad \forall x \in \mathcal{A}_n^2 \quad \textit{and} \quad \int_{\partial B_{t_n \sigma_n^{-\frac{1}{2}}}(x_n)} |\nabla u_n|^2 dx \leq C \sigma_n^{-\frac{N-3}{2}},$$

where $\mathcal{A}_n^2$ is defined in Section 3.

Proof. The proof is the same as those of Proposition 3.1 and Corollary 4.1 in [11], we omit its details here. ∎

Proof of Proposition 5.1. Similar to the proof of Lemma 6.2 in [11], we also can select a concentrating subsequence of $\{u_p\}$, denoted by $\{u_n\}$ with $p = p_n < 2^*$, $p_n \to 2^*$. Hence, it is sufficient to prove that the bubbles $(\sigma_n^i)^{\frac{N-2}{2}} U(\sigma_n^i(x - x_n^i))$ $(1 \leq i \leq k)$ in (5.3) will not appear in the decomposition of u_n. Since the proof is similar to that of Lemma 6.1 in [11], here we only give a sketch of it. Set $\sigma_n = \sigma_n^i$, $x_n = x_n^i$. Then we have the following local Pohozaev identity for $\{u_n\}$ on $B_n = B_{t_n \sigma_n^{-1/2}}(x_n) \cap \Omega$

(5.4)

$$\begin{aligned}\left(\frac{N}{p_n} - \frac{N-2}{2}\right) \int_{B_n} |u_n|^{p_n} dx + t\left(\frac{N}{q+1} - \frac{N-2}{2}\right) \int_{B_n} |u_n|^{q+1} dx \\ = \frac{1}{p_n} \int_{\partial B_n} |u_n|^{p_n} (x - x_0) \cdot \nu d\sigma + \frac{t}{q+1} \int_{\partial B_n} |u_n|^{q+1} (x - x_0) \cdot \nu d\sigma \\ + \int_{\partial B_n} (\nabla u_n \cdot (x - x_0))(\nabla u_n \cdot \nu) d\sigma - \frac{1}{2} \int_{\partial B_n} |\nabla u_n|^2 (x - x_0) \cdot \nu d\sigma \\ + \frac{N}{2^*} \int_{\partial B_n} \nabla u_n \cdot \nu u_n d\sigma,\end{aligned}$$

where ν is the outward normal to ∂B_n.

As in Section 3, set $\partial B_n = \partial_i B_n \cup \partial_e B_n$, where $\partial_i B_n = \partial B_n \cap \Omega$, $\partial_e B_n = \partial\Omega \cap \overline{B_n}$. As in [11], if $\partial_e B_n = \emptyset$, we take x_0 in (5.4) equal to the concentration point x_n; if $\partial_e B_n \neq \emptyset$, we take x_0 out of Ω such that

$$d(x_0, x_n) \leq 2t_n \sigma_n^{-\frac{1}{2}} \quad \text{and} \quad \text{for all } x \in \partial_e B_n, \quad \nu \cdot (x - x_0) < 0. \tag{5.5}$$

Let $B_n' = B_{\sigma_n^{-1}}(x_n)$ and $u_n = u_n^0 + u_n^1 + u_n^2$, where

$$u_n^1 = u_\infty, \quad u_n^2 = \sum_{i=1}^k (\sigma_n^i)^{\frac{N-2}{2}} U(\sigma_n^i(x - x_n^i)), \quad u_n^0 = u_n - u_n^1 - u_n^2.$$

Then we deduce for n large enough

$$\begin{aligned} &\int_{B_n\cap\Omega} |u_n|^{q+1} dx \\ &\quad \geq \int_{B_n'} |u_n|^{q+1} dx \geq \int_{B_n'} \Big(\frac{1}{2}|u_n^2|^2 - 2|u_n^1|^2 - 2|u_n^0|^2\Big)^{\frac{q+1}{2}} \\ &\quad \geq \Big(\frac{1}{2}\Big)^{\frac{q+1}{2}} \int_{B_n'} |u_n^2|^{q+1} dx - 2^{\frac{q+1}{2}} \int_{B_n'} |u_n^1|^{q+1} dx - 2^{\frac{q+1}{2}} \int_{B_n'} |u_n^0|^{q+1} dx. \end{aligned} \tag{5.6}$$

After a direct calculation, we have

$$\int_{B_n'} |u_n^2|^{q+1} dx \geq C\sigma_n^{\frac{(N-2)(q+1)}{2} - N}, \tag{5.7}$$

$$\int_{B_n'} |u_n^1|^{q+1} dx \leq C\sigma_n^{-N}, \tag{5.8}$$

$$\int_{B_n'} |u_n^0|^{q+1} dx \leq C\|u_n^0\|_{L^{2^*}(\Omega)}^{q+1} \sigma_n^{\frac{(N-2)(q+1)}{2} - N}. \tag{5.9}$$

Note that $\|u_n^0\|_{L^{2^*}(\Omega)} \to 0$ as $n \to \infty$. Inserting (5.7)–(5.9) into (5.6), we get for n large enough

$$\int_{B_n\cap\Omega} |u_n|^{q+1} dx \geq C\sigma_n^{\frac{(N-2)(q+1)}{2} - N}. \tag{5.10}$$

As in [11], we only need to consider the right hand side of (5.4) on $\partial_i B_n$. From Lemma 5.3, we get

(5.11)
$$\begin{aligned}\frac{1}{p_n}\int_{\partial_i B_n}|u_n|^{p_n}(x-x_0)\cdot\nu d\sigma+\frac{t}{q+1}\int_{\partial_i B_n}|u_n|^{q+1}(x-x_0)\cdot\nu d\sigma\\+\int_{\partial_i B_n}(\nabla u_n\cdot(x-x_0))(\nabla u_n\cdot\nu)d\sigma-\frac{1}{2}\int_{\partial_i B_n}|\nabla u_n|^2(x-x_0)\cdot\nu d\sigma\\+\frac{N}{2^*}\int_{\partial_i B_n}\nabla u_n\cdot\nu u_n d\sigma\\ \le C\int_{\partial_i B_n}|(x-x_0)\cdot\nu|d\sigma+\int_{\partial_i B_n}|\nabla u_n|^2|x-x_0|d\sigma\\+\left(\int_{\partial_i B_n}|\nabla u_n|^2 d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial_i B_n}|u_n|^2 d\sigma\right)^{\frac{1}{2}}\\ \le C\sigma_n^{-\frac{N-2}{2}}.\end{aligned}$$

Note that $\left(\frac{N}{p_n}-\frac{N-2}{2}\right)\int_{B_n}|u_n|^{p_n}dx>0$, for $p_n\in(2,2^*)$. Inserting (5.10), (5.11) into (5.4), we obtain

$$\sigma_n^{\frac{(N-2)(q+1)}{2}-N}\le C(t)\sigma_n^{-\frac{N-2}{2}},$$

which is a contradiction for n large enough due to $4/(N-2)<q<1$. Thus all of the bubbles in (5.3) can not appear. ∎

The corresponding energy functionals of (1.7), (5.1) are defined as the following respectively:

$$J_t(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{1}{2^*}\int_\Omega|u|^{2^*}dx-\frac{t}{q+1}\int_\Omega|u|^{q+1}dx\quad u\in H_0^1(\Omega).$$

and

$$J_t^{(p)}(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{1}{p}\int_\Omega|u|^{p}dx-\frac{t}{q+1}\int_\Omega|u|^{q+1}dx\quad u\in H_0^1(\Omega)$$

Set

$$\tilde{c}_k:=\inf_{\gamma\in\Gamma_k}\max_{u\in B_k}J_t(\gamma(u)),\quad \tilde{c}_k^n:=\inf_{\gamma\in\Gamma_k}\max_{u\in B_k}J_t^{(p_n)}(\gamma(u)),$$

$$\tilde{b}_k^n:=\inf_{u\in N_k}J_t^{(p_n)}(u),\quad \Gamma_k:=\{\gamma\in C(B_k,H_0^1(\Omega)):\gamma|_{\partial B_k}=\mathrm{id}\},$$

where $B_k,\ N_k$ are given in section 3, $p_n\in(2,2^*)$ and $p_n\to 2^*$ as $n\to\infty$.

Lemma 5.4: *For any positive integer k, $\lim_{n\to\infty}\tilde{c}_k^n=\tilde{c}_k$.*

Proof. For any $v \in H_0^1(\Omega)$,

$$J_t(v) = J_t^{(p_n)}(v) + \frac{1}{p_n}\int_\Omega |v|^{p_n}\,dx - \frac{1}{2^*}\int_\Omega |v|^{2^*}\,dx.$$

Thus, we infer

$$J_t^{(p_n)}(v) \le J_t(v) + \left|\frac{1}{p_n}\int_\Omega |v|^{p_n}\,dx - \frac{1}{2^*}\int_\Omega |v|^{2^*}\,dx\right|, \tag{5.12}$$

and

$$J_t(v) \le J_t^{(p_n)}(v) + \left|\frac{1}{p_n}\int_\Omega |v|^{p_n}\,dx - \frac{1}{2^*}\int_\Omega |v|^{2^*}\,dx\right|. \tag{5.13}$$

Note that id $\in \Gamma_k$, we deduce from (5.12), (5.13) that

(5.14)

$$\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} J_t^{(p_n)}(\gamma(u)) \le \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} J_t(\gamma(u)) + \sup_{u\in B_k}\left|\frac{1}{p_n}\int_\Omega |u|^{p_n}\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx\right|,$$

and

(5.15)

$$\inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} J_t(\gamma(u)) \le \inf_{\gamma\in\Gamma_k}\sup_{u\in B_k} J_t^{(p_n)}(\gamma(u)) + \sup_{u\in B_k}\left|\frac{1}{p_n}\int_\Omega |u|^{p_n}\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx\right|.$$

Since B_k is compact and the functionals

$$H^{(p_n)}(u) = \left|\frac{1}{p_n}\int_\Omega |u|^{p_n}\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx\right|$$

are equicontinuous on B_k, we derive $\lim_{n\to\infty}\sup_{u\in B_k} H^{(p_n)}(u) \to 0$. Therefore, from (5.14), (5.15), we conclude $\lim_{n\to\infty} \tilde{c}_k^n = \tilde{c}_k$. ∎

Lemma 5.5: $\lim\limits_{k\to\infty} \tilde{c}_k = +\infty$.

Proof. It follows from Lemma 5.4 that for every k, there exists an $n_k > k$ such that

$$\left|\tilde{c}_k^{n_k} - \tilde{c}_k\right| < 1/k. \tag{5.16}$$

If we assume that $\lim_{k\to\infty} \tilde{c}_k = c < \infty$. Then from (5.16), we infer

$$\lim_{k\to\infty} \tilde{c}_k^{n_k} = \lim_{k\to\infty} \tilde{c}_k = c. \tag{5.17}$$

Let $\epsilon_0 \in (0, \lambda_1)$ be a fixed number. Define

$$\beta_k := \inf_{u\in Z_k,\ \|u\|_{L^{p_{n_k}}(\Omega)}=1} \int_\Omega (|\nabla u|^2 - \epsilon_0|u|^2)dx.$$

We claim that, up to a subsequence, $\beta_k \to +\infty$ as $k \to \infty$. In fact, since $p_{n_k} < 2^*$, we infer that β_k can be achieved by a function $v_k \in Z_k$, $\int_\Omega |v_k|^{p_{n_k}} dx = 1$, which satisfies

$$-\Delta v_k = \beta_k |v_k|^{p_{n_k}-1} v_k + \epsilon_0 v_k.$$

If $\beta_k \nrightarrow \infty$ as $k \to \infty$, then $\int_\Omega |\nabla v_k|^2 dx \le C$ by the choice of ϵ_0. By using Theorem 1.1 in [11], we conclude that

$$\sup_k \|v_k\|_{L^\infty(\Omega)} \le C. \tag{5.18}$$

Since $v_k \in Z_k$, up to a subsequence, we may assume that

$$v_k \rightharpoonup 0 \quad \text{weakly in} \quad H_0^1(\Omega); \qquad v_k \to 0 \quad \text{a.e. on } \Omega.$$

By (5.18) and the dominated convergence theorem, we deduce that

$$\lim_{k\to\infty} \int_\Omega |v_k|^{p_{n_k}} dx = 0,$$

which is a contradiction due to $\int_\Omega |v_k|^{p_{n_k}} dx = 1$. Thus $\beta_k \to \infty$ as $k \to \infty$. Note that $q \in (0,1)$, by Young inequality, we have for any $u \in Z_k$

$$J_t^{(p_{n_k})}(u) \ge C_5 \|u\|^2_{H_0^1(\Omega)} - C_6 \alpha_k^{-p_{n_k}/2} \|u\|^{p_{n_k}}_{H_0^1(\Omega)} - C_7.$$

Choosing $r_k = \big(\frac{2C_5 \beta_k^{p_{n_k}/2}}{C_6 p_{n_k}}\big)^{1/(p_{n_k}-2)}$, we obtain that if $u \in Z_k$ and $\|u\|_{H_0^1(\Omega)} = r_k$,

$$J_t^{(p_{n_k})}(u) \ge C_5 \Big(1 - \frac{2}{p_{n_k}}\Big) \Big(\frac{2C_5 \beta_k^{p_{n_k}/2}}{C_6 p_{n_k}}\Big)^{2/(p_{n_k}-2)} - C_7. \tag{5.19}$$

Since we have proved that $\beta_k \to \infty$ as $k \to \infty$, from (5.19), we infer that $\tilde{b}_k^{n_k} \to \infty$ as $k \to \infty$. It follows from Theorem 3.5 in [21] that $\tilde{c}_k^{n_k} \ge \tilde{b}_k^{n_k}$, and so $\lim_{k\to\infty} \tilde{c}_k = \lim_{k\to\infty} \tilde{c}_k^{n_k} = +\infty$. ∎

Proof of Theorem 1.4. Let u_k^n be a critical point of $J_t^{(p_n)}$ at level $\tilde{c}_k^n$. By Lemma 5.4, we deduce that $\{u_k^n\}_{n=1}^\infty$ is bounded in $H_0^1(\Omega)$. Then by Proposition 5.1, we can find a subsequence which strongly converges to a solution u_k of (1.7) in $H_0^1(\Omega)$ at level $\tilde{c}_k$. By Lemma 5.5, we obtain infinitely many solutions of (1.7) with positive energy. ∎

Acknowledgement. The author is grateful to the anonymous referee for valuable comments on this paper. This work is partially supported by the Key Project of the National Natural Science Foundation in China No. 10631030.

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